MATHEMATICS

B.Sc, Part II (Paper III)

Topic - Convergency and divergency of series

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Theorem: State and prove Cauchy's Root (or n^{th} root) test for convergence of infinite series.

Statement A series $\sum u_n$ of positive terms is convergent if $u_n^{\frac{1}{n}} < k < 1$ where k is a fixed number and n has any value. And the series is divergent if $u_n^{\frac{1}{n}} \geq 1$ for all n.

Proof When $u_n^{\frac{1}{n}} < k < 1$ for all n, then $u_n < k^n$.

$$\therefore u_1 + u_2 + u_3 + \cdots < k + k^2 + k^3 + \cdots$$

As $k < 1$, the series on the right is convergent and hence the series $u_1 + u_2 + u_3 + \cdots$ is convergent.

When $u_n^{\frac{1}{n}} \geq 1$, then for all n $u_n \geq 1$.

$\therefore u_1 + u_2 + \cdots + u_n \geq 1 + 1 + \cdots$ to n terms $= n$.

$\therefore u_1 + u_2 + u_3 + \cdots \geq n$. When n tends to ∞, the series $\sum u_n$ is divergent.

EXAMPLES

1) Test the convergence of the series whose general term is $\dfrac{n^{n^2}}{(n+1)^{n^2}}$.

Solution Let the given general term of the series be denoted by u_n.

Then $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}} = \left(\dfrac{n}{n+1}\right)^{n^2} = \dfrac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$;

or $u_n^{\frac{1}{n}} = \dfrac{1}{\left(1+\frac{1}{n}\right)^{n^2 \cdot \frac{1}{n}}} = \dfrac{1}{\left(1+\frac{1}{n}\right)^n}$

or $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \dfrac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \dfrac{1}{e} < 1$, as $2 < e < 3$.

Hence by Cauchy's n^{th} root test, the given series is convergent.

2.) Test the series for convergence:

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots \text{to } \infty$$

Solution. Let the n^{th} term of the given series be denoted by u_n.

Then $u_n = \left[\dfrac{(n+1)^{n+1}}{n^{n+1}} - \dfrac{n+1}{n}\right]^{-n}$

or $u_n^{\frac{1}{n}} = \left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$

or $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right) - \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right]^{-1}$

$= [e-1]^{-1} = \dfrac{1}{e-1} < 1$, as $2 < e < 3$.

Hence by Cauchy's n^{th} root test, the given series is convergent.

Theorem Leibnitz Test:

Prove that the alternating series

$\sum_{1}^{\infty}(-1)^{n-1}u_n$ i.e., $u_1-u_2+u_3-u_4+\cdots$ to ∞ is convergent if

(i) each term is numerically less than the preceding i.e., $|u_{n+1}|<|u_n|$, for all n, and

(ii) $u_n \to 0$ as $n \to \infty$.

Proof: Let $S_{2n}=(u_1-u_2)+(u_3-u_4)+\cdots+(u_{2n-1}-u_{2n})$; --- (A)

and this can be written in the form

$S_{2n}=u_1-(u_2-u_3)-(u_4-u_5)-\cdots-(u_{2n-2}-u_{2n-1})-u_{2n}$ --- (B)

since $u_1>u_2>u_3>\cdots$ therefore

$u_1-u_2, u_3-u_4, \cdots$ all are positive.

Also, $u_2-u_3, u_4-u_5, \cdots$ all are positive.

from (A) it, therefore, follows that S_{2n} is positive and increases with n, and from (B) it is seen that S_{2n} is always less than the fixed number u_1.

Hence S_{2n} tends to a limit which is less than u_1.

Again $S_{2n+1}=S_{2n}+u_{2n+1}$.

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}.$$

Since $\lim_{n\to\infty} u_{2n+1}=0$,

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}.$$

The series is therefore convergent, its sum is positive and less than u_1.

Examples

(1) Examine the convergence of the series

$$\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots \text{ to } \infty$$

Solution : We have $\log 2 > 0, \log 3 > 0, \log 4 > 0$, etc.

The terms of the given series are alternately positive and negative. $\because \log 2 < \log 3 < \log 4 < \log 5 < \cdots$

$$\therefore \frac{1}{\log 2} > \frac{1}{\log 3} > \frac{1}{\log 4} > \frac{1}{\log 5} > \cdots$$

That is, $u_n > u_{n+1}$ (numerically), for all $n \in N$.

Therefore each term of the given series is numerically less than the proceding term.

Again u_n (numerically) $= \frac{1}{\log(n+1)}$

or $\lim_{n \to \infty} u_n(\text{numerically}) = 0$

We find that the given series satisfies all the conditions of Leibnitz's theorem. Hence the given series is convergent.

(2) Find whether the following series is convergent:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Solution

Here $u_n = \frac{1}{2n-1}$; $\therefore u_n \to 0$ as $n \to \infty$

and $\frac{u_n}{u_{n-1}} = \frac{2(n-1)-1}{2n-1} = \frac{2n-3}{2n-1} < 1$; $\therefore u_n < u_{n-1}$.

$\therefore$ In this alternating series $u_n \to 0$ as $n \to \infty$ and each term is less than the ~~preced~~ proceding term. So it is convergent.